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Some non-finitely presented Lie algebras

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Abstract

Let L be a free Lie algebra over a field k, I a non-trivial proper ideal of L, $n > 1$ an integer. The multiplicator $H_2(L/I^n, k)$ of L/I^n is not finitely generated, and so in particular, L/I^n is not finitely presented, even when L/I is finite dimensional. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

If R is a free associative algebra, over a field, and I is a two-sided ideal of R, then Lewin [5] proved that I^2 is not finitely generated (as a two-sided ideal!) when the algebra R/I is infinite dimensional. In other words, R/I^2 is not finitely presented in this case. On the other hand, it is easy to see that when R is finitely generated and R/I is finite dimensional, so is R/I^2, and hence I^2 is finitely generated.

Similar behavior is seen in groups. If F is a finitely generated free group, and R is a normal subgroup, then R' is normally finitely generated if, and only if, F/R is finite. In fact, Baumslag et al. proved [1] a stronger fact. Denoting the mth member of the lower central series by γ_m, they proved that for $m > 1$ the Schur multiplier of $F/\gamma_m R$, $H_2(F/\gamma_m R, \mathbb{Z})$, is not finitely generated (as an abelian group) if F/R is not finite.

We note that for the three statements,

(a) R is normally finitely generated,

(b) R/R' is finitely generated as a module over $G = F/R$,

(c) $H_2(G, \mathbb{Z})$ is finitely generated as an abelian group

we have (a) $\Rightarrow$ (b) $\Rightarrow$ (c).

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In this paper we prove a result of similar nature for Lie algebras.

Theorem 1.1. *Let L be a free Lie algebra with basis X, over a field k, and I be* any *non-zero proper ideal of L; then $I' = [I, I]$ is not finitely generated as an ideal. In fact, the "Schur multiplier" of L/I^n, $H_2(L/I^n, k)$, is not finitely generated if $n > 1$, and hence L/I^n is not finitely presented.*

Here I^n denotes I, if $n = 1$, and $[I^{n-1}, I]$ if $n > 1$. Our proof closely follows the lines of [1].

In Section 2 we define some notations and the Magnus embedding. In Section 3 we build a mapping from the Schur multiplier into a tensor product of $n-1$ copies of $U(L/I)$. This is similar to the mapping defined in [1]. In Section 4 we build a specific isomorphism of Hopf modules, keeping in mind that the enveloping algebra of a Lie algebra is a Hopf algebra. In Section 5 we employ the mapping and show that the image of the "Schur multiplicator" is not finite dimensional, thus proving the theorem.

2. Preliminaries and notations

Let $\mathscr{G}$ be a Lie algebra. We will denote the Lie multiplication of two elements $a, b \in \mathscr{G}$ by $[a,b]$. As we will also be considering the enveloping algebra of $\mathscr{G}$, the multiplication in $U(\mathscr{G})$ will be denoted simply as ab, while the action of an element $l \in U(\mathscr{G})$ on an element $a \in \mathscr{G}$ will be denoted by $a \cdot l$. Note that the action is the adjoint action, so that if $l \in L$ then $a \cdot l = [a, l]$.

Let $\mathscr{G}$ be a Lie algebra over a field k, $U(\mathscr{G})$ its enveloping algebra, $\delta U(\mathscr{G})$ the augmentation ideal of U. Suppose $0 \to I \to L \to \mathscr{G} \to 0$ is a free presentation of $\mathscr{G}$, where L is the free Lie algebra with basis X. The enveloping algebra, $U(L)$, is therefore a free associative algebra, with basis X, and $\delta U(L)$ is a free $U(L)$ module, with a basis in one-to-one correspondence with X. Note that over a field, if $\mathscr{G} \neq 0$, $U(\mathscr{G})$ is infinite dimensional, and is without zero divisors.

In addition, if $\mathscr{G}$ is a Lie algebra over a field and $U(\mathscr{G})$ is its enveloping algebra, let $U_n(\mathscr{G})$ be the subspace of $U(\mathscr{G})$ spanned by all the products of at most n factors from $\mathscr{G}$. This gives a well-known ascending filtration of $U(\mathscr{G})$, and we can define the *degree* of an element l to be the *least* integer n such that $l \in U_n(\mathscr{G})$. This function has the properties:

(1) $\deg(a+b) \leq max\{\deg(a), \deg(b)\}$,

(2) if $\deg(a) < \deg(b)$ then $\deg(a+b) = \deg(b)$,

(3) $\deg(ab) = \deg(a) + \deg(b)$.

In particular, if $x \in \mathscr{G}$ is non-zero then the degree of x is 1, so if $x_1, x_2, \ldots, x_n \in \mathscr{G}$ are all non-zero then $\deg(x_1 x_2 \cdots x_n) = n$.

Via the adjoint action, I/I' carries the structure of a $U(L)$ module, and I acts trivially. All modules will be right modules. Therefore, I/I' is a $U(L/I)$ module in a natural way. There is a well-known embedding of $U(L/I)$ modules, the Magnus embedding, described below, of I/I' into $\delta U(L)\otimes_{U(L)} U(L/I)$. This embedding will be denoted by $\phi : I/I' \to \delta U(L)\otimes_{U(L)} U(L/I)$. The action of L on $\delta U(L)\otimes_{U(L)} U(L/I)$ is by right multiplication on the right-hand term.

The embedding can be defined in the following way. First define $\phi : I \to \delta U(L)\otimes_{U(L)} U(L/I)$ by $\phi(x) = x\otimes 1$. By using the Poincare–Birkhoff–Witt theorem, and the structure it gives to $U(L)$, it can be seen that this is a mapping of $U(L)$ modules, i.e. $\phi(a\cdot l) = \phi(a)l$. First we check the statement for elements of L. If $l \in L$ then $a\cdot l = [a,l]$ and $\phi([a,l]) = [a,l]\otimes 1 = (al - la)\otimes 1 = a\otimes l - l\otimes a$. However, $a = 0$ in $U(L/I)$ so $\phi([a,l]) = a\otimes l = (a\otimes 1)l = \phi(a)l$. Consider now the subalgebra $A = \{u \in U(L) | \phi(x\cdot u) = \phi(x)u \ \forall x \in I\}$. Since $L \subset A$ then $A = U(L)$, thus ϕ is a $U(L)$ module homomorphism.

It is left to show that $\ker\phi = I'$. If $x \in I'$ then x can be written as $x = \sum [a_i, b_i]$, $a_i, b_i \in I$, so that $\phi(x) = x\otimes 1 = \sum [a_i,b_i]\otimes 1 = \sum (a_ib_i - b_ia_i)\otimes 1 = \sum a_i\otimes b_i - b_i\otimes a_i$. Since $a_i, b_i \in I$ then their images in $U(L/I)$ are 0 so that $\phi(x) = 0$. Therefore, $I' \subset \ker\phi$. On the other hand, suppose $x \in \ker\phi$. Since $\delta U(L)$ is a free $U(L)$ module with basis $\{x_i\}$ where x_i is a basis of L as a free Lie algebra, we have $x\otimes 1 = \sum x_i\otimes f_i$, where, since $\phi(x) = 0$, $f_i = 0$ in $U(L/I)$. Let us denote by $\tilde{I}$ the kernel of the mapping $U(L) \to U(L/I)$, so that $f_i \in \tilde{I}$. But $\tilde{I} = U(L)I = IU(L)$ and thus by the Poincare–Birkhoff–Witt theorem this kernel is a free left and right $U(L)$ module with a basis that is a basis of I as a subalgebra of L. Therefore, $f_i = \sum w_{i,j}a_j$ where a_j are a basis of I. It follows that $x = \sum x_i w_{i,j} a_j$. Consider now the image of x,$\bar{x}$, in I/I'. Since I/I' is the commutative Lie algebra with a basis that is a basis of I as a subalgebra of L, then $\bar{x} = \sum \lambda_j a_j$, where $\lambda_j \in k$. In other words, $x = \sum \lambda_j a_j + w$, $w \in I'$. But since $I' \subset \ker\phi$ then we can assume $x = \sum \lambda_j a_j$. On the other hand, $\phi(x) = 0$ so $x = \sum x_i w_{i,j} a_j$. Since $\tilde{I}$ is a free $U(L)$ module with basis a_i we have $\lambda_j = \sum x_i w_{i,j}$, but $x_i \in \delta U(L)$, so $\lambda_j = 0$. Hence, $x \in I'$, therefore $\ker\phi = I'$.

Another proof of the fact that $\ker\phi = I'$ can be found in [2, Section 8] as the Magnus embedding is a special case of the derivations defined there.

Throughout the remainder of this paper I will be a proper non-zero ideal of L, and $n > 1$ will be an integer.

3. An image of $H_2(L/I^n, k)$

Consider $H_2(L/I^n, k)$. It is known (e.g. [7, p. 233]) that the analogue of the Hopf formula for groups holds for Lie algebras. Therefore,

$$H_2(L/I^n, k) = I^n/[I^n, L] = (I^n/I^{n+1})\otimes_{U(L)} k.$$

We know from the Širšov–Witt theorem (see e.g. [6, p. 44]) that I is a free Lie algebra. Hence I^n/I^{n+1} is, in a natural way, identifiable with the nth homogeneous component of the free Lie algebra with basis that is a basis of I/I' as a vector space. Since the free Lie algebra of a free module can be embedded in the tensor algebra over this module, the nth homogeneous component can be embedded into the n-fold tensor product, i.e. I^n/I^{n+1} can be embedded in $\otimes^n I/I'$, where the tensor is over k. Any unadorned tensor product below is to be taken to be over k. We need this embedding to be a $U(L/I)$ module homomorphism, and it is easy to see that this is indeed the case when $U(L/I)$ acts on I^n/I^{n+1} via the adjoint action, and on $\otimes^n I/I'$ diagonally. The module $\otimes^n I/I'$ can again can be embedded, through the Magnus embedding, into

$$\bigotimes^n(\delta U(L) \otimes_{U(L)} U(L/I)).$$

Tensoring this with k over L we get a mapping

$$H_2(L/I^n,k) \approx \bigotimes^n I/I' \otimes_{U(L)} k \to \bigotimes^n(\delta U(L) \otimes_{U(L)} U(L/I)) \otimes_{U(L)} k.$$

Since $\delta U(L)$ is a free $U(L)$ module, with a basis X that is a basis of L as a Lie algebra, we can define for each $x \in X$ a projection, denoted $p_x : \delta U(L) \otimes_{U(L)} U(L/I) \to U(L/I)$. We therefore have for each n-tuple $(x_1,x_2,\dots,x_n) \in X^n$ a mapping $\phi_{x_1,\dots,x_n} := (p_{x_1} \otimes \cdots \otimes p_{x_n} \otimes 1) \circ \phi$

$$\phi_{x_1,x_2,\dots,x_n} : H_2(L/I^n,k) \to \bigotimes^n U(L/I) \otimes_{U(L)} k.$$

Since $I/I' \to U(L/I) \otimes \delta U(L)$ is an embedding, there exist elements $\alpha \in I/I'$ and $x \in X$ such that under the Magnus embedding and the projection by x the image $a = \phi_x(\alpha)$ is non-zero. These elements will be put to use below.

4. Isomorphism of Hopf modules

As seen in the previous section the image of the multiplicator lies in $(U(L/I) \otimes U(L/I) \cdots \otimes U(L/I)) \otimes_{U(L)} k$. On the other hand, it is well known that the enveloping algebra is a Hopf algebra, and the action with which this module is endowed is consistent with the standard Hopf structure on $U(L/I)$, which is the diagonal action. We shall use the following notation for the structure of Hopf algebras and modules. Let H be a Hopf algebra and M a Hopf module over H. The diagonal mapping of H will be denoted by Δ, and the n-fold application of Δ by Δ_n (by the co-associativity of H the components on which we apply Δ each time do not matter). The co-unit of H will be denoted by ε (also sometimes known as the augmentation). The antipode map of H will be denoted by S. The usual action of H on M will be denoted by multiplication on the right, and the co-action of M will be denoted by ρ. If $h \in H$ then $\Delta(h)$ will be written as $\Delta(h) = \sum_{i=1}^{l} h_1^i \otimes h_2^i$, and $\Delta(h_1^i) = \sum_{j=1}^{l(i)} h_{1,1}^{i,j} \otimes h_{1,2}^{i,j}$. If $m \in M$ then $\rho(m) = \sum m_0^i \otimes m_1^i$.

It is known (see e.g. [4, p.15]) that for any Hopf algebra H and Hopf module M, $M \approx M' \otimes H$, where $M' = \{m \in M | \rho(m) = m \otimes 1\}$ with the isomorphism $m \mapsto \sum m_0^i \cdot S(m_{1,1}^{i,j}) \otimes m_{1,2}^{i,j}$, where this is actually a double sum on both i and j. It should also be noted that $M' \otimes H$ is a trivial Hopf module, i.e. one for which $(m \otimes h)l = m \otimes hl$.

If we now also tensor with k over H we will get

$$M \otimes_H k \approx (M' \otimes H) \otimes_H k.$$

However, since $M' \otimes H$ is a trivial (in the sense defined above) Hopf module we get

$$M \otimes_H k \approx (M' \otimes H) \otimes_H k \approx M' \otimes (H \otimes_H k) \approx M'.$$

The isomorphism is

$$m \otimes 1 \mapsto \sum m_0^i \cdot S(m_{1,1}^{i,j}) \otimes m_{1,2}^{i,j} \otimes 1 \mapsto \sum m_0^i \cdot S(m_{1,1}^{i,j})\varepsilon(m_{1,2}^{i,j}) = \sum m_0^i \cdot S(m_1^i).$$

If we take $M = W \otimes H$ with W any Hopf module, H acting with the diagonal action and

$$\rho(w \otimes h) = w \otimes \Delta(h),$$

then $M' = W \otimes k \approx W$. In this case, if $m = w \otimes h$ then $\rho(w \otimes h) = w \otimes \Delta(h)$ so $m_0^i = w \otimes h_1^i$ and $m_1^i = h_2^i$. Therefore, the explicit form of the isomorphism is

$$w \otimes h \otimes 1 \mapsto \sum (w \otimes h_1^i)\Delta(S(h_2^i)).$$

However, we know that the image is in M', so we can apply $1 \otimes \varepsilon$ to the image and not change it. Also if $h \in H$ then from the definition of a Hopf algebra $(1 \otimes \varepsilon)(\Delta(h)) = h \otimes 1$.

Therefore, the image is

$$\begin{aligned}
&(1 \otimes \varepsilon)\left[\sum (w \otimes h_1^i)\Delta(S(h_2^i))\right] \\
&\quad = \sum (w \otimes \varepsilon(h_1^i))[(1 \otimes \varepsilon)(\Delta(S(h_2^i)))] = \sum (w \otimes 1)(\varepsilon(h_1^i)S(h_2^i) \otimes 1) \\
&\quad = (w \otimes 1)(S(h) \otimes 1),
\end{aligned}$$

so the image in W is

$$w \otimes h \otimes 1 \mapsto wS(h).$$

In our case we are interested in the module $\bigotimes^n H$, so we can take $W = \bigotimes^{n-1} H$ and the isomorphism will be

$$h_1 \otimes h_2 \otimes \cdots \otimes h_n \otimes 1 \mapsto (h_1 \otimes h_2 \otimes \cdots \otimes h_{n-1})\Delta_{n-1}(S(h_n)).$$

5. Computations

We can now prove Theorem 1.1, i.e. show that $H_2(L/I^n, k)$ is not finitely generated by exhibiting an infinite number of elements of the multiplicator, whose images in $\bigotimes^{n-1} U(L/I)$ are linearly independent. We shall deal with several cases. In each of them we shall construct elements of $H_2(L/I^n,k)$ that have one parameter l, where $l \in U(L/I)$. In other words, we shall construct a k-linear map $f : U(L/I) \to H_2(L/I^n,k) \to \bigotimes^{n-1} U(L/I)$. It is obviously enough to show that $\ker f = k \cdot 1$ (since $U(L/I)$ is not finite dimensional). In other cases, we shall show that $\operatorname{Im} f$ is not finite dimensional by proving that it has elements of unbounded degree.

Recall the elements $\alpha \in I/I'$ and $x \in X$ such that $a = \phi_x(\alpha)$ was non-zero, and consider all elements of the form $[\alpha \cdot l, \alpha, \dots, \alpha] \otimes 1$, where l is any element of $\delta U(L/I)$. Obviously, this element is in I^n. Its image, using the mapping $\phi_{x,x,\dots,x}$ will be $[al, a, \dots, a] \otimes 1$. In other words, $f(l) = [al, a, \dots, a] \otimes 1$. Note that if $l \in k \cdot 1$ then $f(l) = 0$ since in that case $[a \cdot l, a] = 0$. An easy induction shows that

$$[a,b,b,\dots,b] \otimes 1 = \sum (-1)^i \binom{n-1}{i} \bigotimes^{i} b \otimes a \bigotimes^{n-1-i} b \otimes 1,$$

where $\bigotimes^i b$ means $b \otimes b \otimes \cdots \otimes b$ (i times). The referee points out that this formula is known as the Cartan–Weyl formula. Therefore, under the Hopf module isomorphism

$$f(l) = \sum (-1)^i \binom{n-1}{i} \left(\bigotimes^{i} a \otimes al \bigotimes^{n-2-i} a \right) \Delta_{n-1}(S(a))$$
$$+(-1)^{n-1} \left(\bigotimes^{n-1} a \right) \Delta_{n-1}(S(al)).$$

But $S(al) = S(l)S(a)$ so $\Delta_{n-1}(S(al)) = \Delta_{n-1}(S(l))\Delta_{n-1}(S(a))$ and hence

$$f(l) = \left[\sum (-1)^i \binom{n-1}{i} \left(\bigotimes^{i} a \otimes al \bigotimes^{n-2-i} a \right) \right.$$
$$\left. +(-1)^{n-1} \left(\bigotimes^{n-1} a \right) \Delta_{n-1}(S(l)) \right] \Delta_{n-1}(S(a)).$$

This can be rewritten as

$$f(l) = (a \otimes a \otimes \cdots \otimes a) \left[\sum (-1)^i \binom{n-1}{i} \left(\bigotimes^{i} 1 \otimes l \bigotimes^{n-2-i} 1 \right) \right.$$
$$\left. +(-1)^{n-1} \Delta_{n-1}(S(l)) \right] \Delta_{n-1}(S(a)).$$

Since $U(L/I)$ is without zero divisors and we are only interested in $\ker f$ or the dimension of $\operatorname{Im} f$, we can consider instead the function

$$f(l) = \sum (-1)^i \binom{n-1}{i} \left(\bigotimes^{i} 1 \otimes l \bigotimes^{n-2-i} 1 \right) + (-1)^{n-1} \Delta_{n-1}(S(l)).$$

In order to compute $\ker f$, we can apply ε to all but the jth coordinate of each monomial. This operator, applied to $\bigotimes^i 1 \otimes l \bigotimes^{n-2-i} 1$, yields $l\delta_{ij}$ (since $\varepsilon(l) = 0$), while applied to $\Delta_{n-1}(S(l))$ yields (because ε is a counit) $S(l)$. Therefore, for each $0 \le j < n$ the result is

$$(-1)^j \binom{n-1}{j} l + (-1)^{n-1} S(l) = 0.$$

Therefore $S(l) = (-1)^{n+j} \binom{n-1}{j} l$.

If $n > 2$ we get $S(l) = (-1)^n l$ and $S(l) = (-1)^{n+1}(n-1)l$.

Therefore $(-1)^n l = (-1)^{n+1}(n-1)l$, i.e.

$$nl = 0.$$

As was mentioned above, there are several cases.

Case I: If $\operatorname{char}(k)$ does not divide n and $n > 2$ then for any $l \in \delta U(L/I)$ we have $f(l) \ne 0$, i.e. $\ker f = k \cdot 1$.

Case II: If $\operatorname{char}(k) \ne 2$. We wish to show that $\operatorname{Im} f$ is not finite dimensional. Denoting by $f_1(l)$ the application of ε to all but the first coordinate, we get $f_1(l) = l + (-1)^{n-1}S(l)$. This is true also when $n = 2$. Since f_1 is simply f composed with another function, obviously $\dim(\operatorname{Im} f_1) \le \dim(\operatorname{Im} f)$. Therefore, it is enough to consider f_1. However, if x is any non-zero Lie element in $U(L/I)$ then $S(x^i) = (-1)^i x^i$. So $f_1(x^i) = x^i + (-1)^{i+n-1}x^i$. Since $\operatorname{char}(k) \ne 2$ then for all i of the correct parity we will have $f_1(x^i) = 2x^i \ne 0$, but $\deg x^i = i$ will be unbounded, so we are finished.

Case III: The only case left is $\operatorname{char}(k) = 2$ and n even. In this case we still have $f_1(l) = l - S(l)$. Suppose L/I is not commutative, therefore there exist $x, y \in L$ such that $[x,y] \notin I$, i.e. $[x,y] \ne 0$ in $U(L/I)$. Consider $l_i = xy^i$. Obviously, $S(l_i) = y^i x$, so $f_1(l_i) = xy^i - y^i x = [x, y^i]$. However, the mapping $u \mapsto [x,u]$ is a derivation of $U(L/I)$, and therefore

$$[x, y^i] = \sum_{j=0}^{i-1} y^j [x,y] y^{i-j-1}.$$

Note that $[x,y]y = y[x,y] + [[x,y],y]$, and hence $y^j[x,y]y^{i-j-1} \equiv y^{i-1}[x,y] \bmod U_{i-1}(L/I)$. Thus, $[x,y^i] \equiv iy^{i-1}[x,y] \bmod U_{i-1}(L/I)$, and if i is odd then $\deg f_1(l_i) = i$. Thus, the degree of the elements of the image is unbounded, so the image is infinite dimensional.

Case IV: There remains the case where L/I is commutative. Thus, if L has basis X, then $L' \subset I$ so $I/L' \subset L/L'$ is a subspace, and we can perform a linear change of basis of L, so that $I = \langle L', X_1 \rangle$, where X_1 is a proper subset of X. Consider the Lie algebra over $\mathbb{Z}$, $L_1 = \langle Y \rangle$, $I_1 = \langle L_1', Y_1 \rangle$, where Y and Y_1 are disjoint copies of X and X_1. We now use the universal coefficient theorem (see e.g. [3, p. 176]) which in our case states that if k is any $\mathbb{Z}$ module then

$$0 \to H_2(L_1/I_1^n, \mathbb{Z}) \otimes_{\mathbb{Z}} k \to H_2(L_1/I_1^n \otimes_{\mathbb{Z}} k, k) \to \operatorname{Tor}_1^{\mathbb{Z}}(H_1(L_1/I_1^n, \mathbb{Z}), k) \to 0$$

is exact. Since $H_1(L_1/I_1^n, \mathbb{Z}) = (L_1/I_1^n)_{ab} = L_1/L_1'$ is a free $\mathbb{Z}$ module then

$$\operatorname{Tor}_1^{\mathbb{Z}}(H_1(L_1/I_1^n, \mathbb{Z}), k) = 0.$$

Take $k = \mathbb{Q}$. We have $H_2(L_1/I_1^n, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \approx H_2(L_1/I_1^n, \otimes_{\mathbb{Z}} \mathbb{Q}, \mathbb{Q})$. However, $L_1/I_1^n \otimes_{\mathbb{Z}} \mathbb{Q}$ is simply L_2/I_2^n where $L_2 = \langle Y \rangle$ and $I_2 = \langle L_2', Y_1 \rangle$ taken over $\mathbb{Q}$. Since $\mathbb{Q}$ has characteristic 0, we know that $H_2(L_1/I_1^n \otimes_{\mathbb{Z}} \mathbb{Q}, \mathbb{Q})$ is infinite dimensional. Therefore, $H_2(L_1/I_1^n, \mathbb{Z})$ must also have infinite torsion-free rank as a $\mathbb{Z}$-module. Apply now the universal coefficient theorem with k any field of characteristic 2. Again $H_2(L_1/I_1^n, \mathbb{Z}) \otimes_{\mathbb{Z}} k \approx H_2(L_1/I_1^n \otimes_{\mathbb{Z}} k, k)$. Once again $L_1/I_1^n \otimes_{\mathbb{Z}} k$ is exactly L/I^n of the original Lie algebra. However, since $H_2(L_1/I_1^n, \mathbb{Z})$ has infinite rank then $H_2(L_1/I_1^n, \mathbb{Z}) \otimes_{\mathbb{Z}} k$ is not finitely generated, thus we have proved Theorem 1.1.

Note that in the case $L = \langle x, y \rangle$, $I = L'$ and k is of characteristic 2, even though $H_2(L/I', k)$ is not finitely generated, the image in $U(L/I)$, under any of the projections, will be 0.

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